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Ground-State Correlation Functions for an Impenetrable Bose Gas with Neumann or Dirichlet Boundary Conditions

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We study density correlation functions for an impenetrable Bose gas in a finite box, with Neumann or Dirichlet boundary conditions in the ground state. We derive the Fredholm minor determinant formulas for the correlation functions. In the thermodynamic limit, we express the correlation functions in terms of solutions of nonlinear differential equations which were introduced by Jimbo, Miwa, Môri, and Sato as a generalization of the fifth Painlevé equations.

KEY WORDS: Solvable model; correlation functions; boundary conditions; Bose gas; Painlevé transcendent.

1. INTRODUCTION

In the standard treatment of quantum integrable systems, one starts with a finite box and imposes periodic boundary conditions, in order to ensure integrability. Recently, there has been increasing interest in exploring other possible boundary conditions compatible with integrability.

With non-periodic boundary conditions, the works on the Ising model are among the earliest. By combinatorial arguments, McCoy and Wu[1] studied the two-dimensional Ising model with a general boundary. They calculated the spin-spin correlation functions of two spins in the boundary row. Using fermions, Bariev[2] studied the two-dimensional Ising model with a Dirichlet boundary. He calculated the local magnetization and derived the third Painlevé differential equations in the scaling limit.

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Bariev[3] generalized his calculation to a general boundary case. In the Neumann boundary case, he also derived the third Painlevé differential equations in the scaling limit. Sklyanin[4] began a systematic approach to open boundary problems, so-called open boundary Bethe Ansatz. Jimbo *et al.*[5] calculated correlation functions of local operators for antiferromagnetic XXZ chains with a general boundary, using Sklyanin's algebraic framework and the representation theory of quantum affine algebras.

Sklyanin[4] explained the integrability of the open boundary impenetrable bose gas model, using boundary Yang Baxter equations. In this paper, we will study density correlation functions (density matrix) for an impenetrable bose gas with Neumann or Dirichlet boundary conditions. Schultz[6] studied field correlation functions for an impenetrable bose gas with priodic boundary conditions. He discretized the second quantized Hamiltonian and found that the discretized Hamiltonian was the isotropic XY model Hamiltonian. He diagonalized the discretized Hamiltonian by introducing fermion operators. Using the N particle ground state eigenvector for the discretized Hamiltonian, Schultz derived an explicit formula of correlation functions for an impenetrable bose gas in the continuum limit. Lenard[7] pointed out that Schultz's formula could be written by Fredholm minor determinants. Therefore this formula is called Schultz-Lenard formula. In this paper, we will derive Schultz-Lenard type formula for Neumann or Dirichlet boundary condition. Following Schltz, we employ two devices. We consider the N particle ground state of the discretized Hamiltonian. We then fermionize the discretized N particle system by using the Jordan-Wigner transformation. In the continuum limit, we derive the Fredholm minor determinant formula for correlation function, which has the integral kernel:

$$\frac{\pi}{2L}\left\{\frac{\sin\dfrac{2N+1}{2L}\pi(x-x')}{\sin\dfrac{1}{2L}\pi(x-x')}+\varepsilon\frac{\sin\dfrac{2N+1}{2L}\pi(x+x')}{\sin\dfrac{1}{2L}\pi(x+x')}\right\} \tag{1.1}$$

(*L: box size, N: the number of particles, $\varepsilon=+$: Neumann, $\varepsilon=-$: Dirichlet*)

Jumbo, Miwa, Môri, and Sato[8] developed the deformation theory for Fredholm integral equation of the second kind with the special kernel $[\sin(x-x')/x-x']$. They introduced a system of nonlinear partial differential equation, which becomes the fifth Painlevé in the simplest case. They showed that the correlation functions without boundaries was the τ-function of their generalization of the fifth Painlevé equations. In this paper, we express the correlation functions for Neumann or Dirichlet boundaries in terms of solutions of Jimbo, Miwa, Môri, and Sato's generalization of the fifth Painlevé equations, hereafter refered to as the JMMS equations. In the thermodynamic limit ($N, L \to \infty, N/L$: *fixed*), we reduce the differential equations for correlation functions with Neumann or Dirichlet boundaries to that without boundaries, using the reflction relation between two integral kernels $[\sin(x-x')/x-x'] + \varepsilon[\sin(x+x')/x+x']$ and $[\sin(x-x')/x-x']$. The two point correlation function with Neumann boundary is described by the Eqs. (2.29) and (2.30). In the case with boundary, the differential equation for the two point correlation function cannot be described by an ordinary differential equation. We need three variable case of the JMMS equations.

Physically, the long distance asymptotics of the correlation function are interesting. The long distance asymptotics of the ordinary differential Painleve V is known. But, for many variable case, the asymptotics of the JMMS equations are not known. Therefore we cannot describe the long distance asymptotics of the correlation functions with boundary in this paper. To evaluate the asymptotics of the solution of the JMMS equation is our future problem.

Now a few words about the organization of the paper. In Section 2, we state the problem and summarize the main results. In Section 3, we derive an explicit formula for the correlation functions in a finite box. In Section 4, we write down the differential equations for the correlation functions in the thermodynamic limit.

2. FORMULATION AND RESULTS

The purpose of this section is to formulate the problem and summarize the main results. The quantum mechanics problem we shall study is defined by the following four conditions. Let $N \in \mathbf{N}$, $(N \geqslant 2)$, $L \in \mathbf{R}$, $\vartheta_0, \vartheta_L \in \mathbf{R}$.

1. The wave function $\psi_{N,L} = \psi_{N,L}(x_1,..., x_N \mid \vartheta_0, \vartheta_L)$ satisfies the free-particle Schrödinger equation for the motion of N particles in one dimension $(0 \leqslant (x_i \neq x_j) \leqslant L$. Here the variables $x_1,..., x_N$ stand for the coordinates of the particles.
2. The wave function $\psi_{N,L}$ is symmetric with respect to the coordinates.

$$\psi_{N,L}(x_1,..., x_N \mid \vartheta_0, \vartheta_L) = \psi_{N,L}(x_{\sigma(1)},..., x_{\sigma(N)} \mid \vartheta_0, \vartheta_L), \qquad (\sigma \in S_N) \tag{2.1}$$

3. The wave function satisfies the open boundary conditions in a box $0 \leqslant x_j \leqslant L$, $(j = 1,..., N)$

$$\left(\frac{\partial}{\partial x_j} - \vartheta_0\right) \psi_{N,L}(x_1,\dots, x_N \mid \vartheta_0, \vartheta_L)\Big|_{x_j=0} = 0, \qquad (j = 1,\dots, N) \tag{2.2}$$

$$\left(\frac{\partial}{\partial x_j} + \vartheta_0\right) \psi_{N,L}(x_1,\dots, x_N \mid \vartheta_0, \vartheta_L)\Big|_{x_j=L} = 0, \qquad (j = 1,\dots, N) \tag{2.3}$$

4. The wave function $\psi_{N,L}$ vanishes whenever the particle coordinates coincide.

$$\psi_{N,L}(x_1,\dots, x_i \dots, x_j,\dots, x_N \mid \vartheta_0, \vartheta_L) = 0, \qquad \text{for} \quad x_i = x_j \tag{2.4}$$

In this paper we shall be concerned with the ground state. The wave functon is given by

$$\psi_{N,L}(x_1,\dots, x_N \mid \vartheta_0, \vartheta_L) = \frac{1}{\sqrt{V_{N,L}(\vartheta_0, \vartheta_L)}} \left| \det_{1 \leqslant j,k \leqslant N} (\lambda_j \cos(\lambda_j x_k) + \vartheta_0 \sin(\lambda_j x_k)) \right| \tag{2.5}$$

Here the momenta $0 < \lambda_1 < \cdots < \lambda_N$ are determined from the boundary condition for $\psi_{N,L}$ which amounts to the equations

$$2L\lambda_j + \theta_{\vartheta_0}(\lambda_j) + \theta_{\vartheta_L}(\lambda_j) = 2\pi j, \qquad (j = 1,\dots, N) \tag{2.6}$$

where we set $\theta_d(\lambda) = i \log(id + \lambda / id - \lambda)$. We take the branch $-\pi < \theta_d(\lambda) \leqslant \pi$, $(d \geqslant 0)$. Here $V_{N,L}(\vartheta_0, \vartheta_L)$ is a normalization factor defined by

$$V_{N,L}(\vartheta_0, \vartheta_L) = \frac{N!}{2^{2N}} \det_{1 \leqslant j,k \leqslant N} \left(\sum_{\varepsilon, \varepsilon' = \pm} (\lambda_j - i\varepsilon\vartheta_0)(\lambda_k - i\varepsilon'\vartheta_0) \int_0^L e^{i(\varepsilon\lambda_j + \varepsilon'\lambda_k) y}\, dy \right). \tag{2.7}$$

The wave function is not translationally invariant and has the normalization,

$$\int_0^L \cdots \int_0^L dy_1 \cdots dy_N\, \psi_{N,L}(y_1 \dots, y_N \mid \vartheta_0, \vartheta_L)^2 = 1 \tag{2.8}$$

The following equation holds for any parameter λ,

$$\left(\frac{\partial}{\partial x} - \vartheta_0\right) (\lambda \cos(\lambda x) + \vartheta_0 \sin(\lambda x))\Big|_{x=0} = 0 \tag{2.9}$$

The following equivalent relation holds,

$$\left(\frac{\partial}{\partial x} + \vartheta_L\right) (\lambda \cos(\lambda x) + \vartheta_0 \sin(\lambda x))\Big|_{x=L} = 0 \Leftrightarrow e^{2iL\lambda} = \frac{(\lambda + i\vartheta_L)(\lambda + i\vartheta_0)}{(\lambda - i\vartheta_L)(\lambda - i\vartheta_0)} \tag{2.10}$$

From (2.9), (2.10) and Girardeau's observation on fermions and impenetrable bosons correspondence in one dimension,[9] we can show that the wave function $\psi_{N,L}$ satisfies the above four conditions. We shall be interested in the correlation functions (density matrix) given by

$$\rho_{n,N,L}(x_1,\ldots,x_n\,|\,x'_1,\ldots,x'_n\,|\,\vartheta_0,\vartheta_L)$$
$$=\frac{(n+N)!}{N!}\int_0^L\cdots\int_0^L dy_{n+1}\cdots dy_{n+N}\psi_{n+N,L}(x_1,\ldots,x_n,y_{n+1},\ldots,y_{n+N}\,|\,\vartheta_0,\vartheta_L)$$
$$\times\psi_{n+N,L}(x'_1,\ldots,x'_n,y_{n+1},\ldots,y_{n+N}\,|\,\vartheta_0,\vartheta_L) \tag{2.11}$$

In this paper, following,[6] we reduce our problem to that of discrete M intervals. Set $\varepsilon=L/(M+1)$. Let $|v_1\rangle=\binom{1}{0}$, $|v_2\rangle=\binom{0}{1}$ be the standard basis of $V=\mathbf{C}^2$. Let $\langle v_i|$, $(i=1,2)$ be the dual basis given by $\langle v_i|v_j\rangle=\delta_{i,j}$, $(i,j=1,2)$. The action of $O\in\mathrm{End}(\mathbf{C}^2)$ on $\langle v_i|$, $(i=1,2)$ is defined by $(\langle v_i|O)\,|v_j\rangle)$, $(j=1,2)$. Set $|\Omega_0\rangle=|v_1\rangle^{\otimes M}$ and $\langle\Omega_0|=\langle v_1|^{\otimes M}$. Set

$$\phi^+=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad \phi=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad \sigma^z=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{2.12}$$

Following the usual convention, we let $\phi_j^+,\phi_j,\sigma_j^z$ signify the operators acting on then jth tensor component of $V^{\otimes M}$. Introduce fermion operators ψ_m^+,ψ_m be the Jordan-Wigner transformation

$$\psi_m^+=\sigma_1^z\cdots\sigma_{m-1}^z\phi_m^+,\qquad \psi_m=\sigma_1^z\cdots\sigma_{m-1}^z\phi_m,\qquad (m=1,\ldots,M) \tag{2.13}$$

The fermion operators have the anti-commutation relations

$$\{\psi_m^+,\psi_n\}=\delta_{m,n},\qquad \{\psi_m,\psi_n\}=\{\psi_m^+,\psi_n^+\}=0 \tag{2.14}$$

Here we use the notation $\{a,b\}=ab+ba$. Set

$$|\Omega_{N,M}(\vartheta_0,\vartheta_L)\rangle$$
$$=\sqrt{\frac{1}{N!}}\sum_{m_1,\ldots,m_N=1}^{M}\psi_{N,L}(\varepsilon m_1,\ldots,\varepsilon m_N\,|\,\vartheta_0,\vartheta_L)\,\phi_{m_1}^+\cdots\phi_{m_N}^+\,|\Omega_0\rangle$$
$$=\frac{1}{\sqrt{N!\,V_{N,L}(\vartheta_0,\vartheta_L)}}\prod_{j=1}^{N}\sum_{m_j=1}^{M}(\lambda_j\cos(\varepsilon m_j\lambda_j)+\vartheta_0\sin(\varepsilon m_j\lambda_j))$$
$$\times\psi_{m_1}^+\cdots\psi_{m_N}^+\,|\Omega_0\rangle \tag{2.15}$$

$$\langle \Omega_{N,M}(\vartheta_0, \vartheta_L)|$$
$$= \sqrt{\frac{1}{N!}} \sum_{m_1,\dots,m_N=1}^{M} \psi_{N,L}(\varepsilon m_1,\dots, \varepsilon m_N \mid \vartheta_0, \vartheta_L)\langle \Omega_0| \phi_{m_1} \cdots \phi_{m_N}$$
$$= \frac{1}{\sqrt{N!\, V_{N,L}(\vartheta_0, \vartheta_L)}} \prod_{j=1}^{N} \sum_{m_j=1}^{M} (\lambda_j \cos(\varepsilon m_j \lambda_j) + \vartheta_0 \sin(\varepsilon m_j \lambda_j))$$
$$\times \langle \Omega_0| \psi_{m_1} \cdots \psi_{m_N} \tag{2.16}$$

Using the above vectors, we can calculate correlation functions in the continuum limit as

$$\rho_{n,N,L}(x_1,\dots, x_n \mid x'_1,\dots, x'_n \mid \vartheta_0, \vartheta_L)$$
$$= \lim_{M\to\infty} \left(\frac{L}{M}\right)^N \langle \Omega_{n+N,M}(\vartheta_0, \vartheta_L)| \phi_{s_1}\phi_{s_2} \cdots \phi_{s_n}\phi^+_{t_1}\phi^+_{t_2} \cdots \phi^+_{t_n}$$
$$\times |\Omega_{n+N,M}(\vartheta_0, \vartheta_L)\rangle \tag{2.17}$$

where we take the limit $M \to \infty$ in such a way that $\varepsilon s_j \to x_j$, $\varepsilon t_j \to x'_j$, (L: *fixed*). The equation (2.17) follows from (2.18) and (2.19).

$$n^2({}_{N+n}C_n)^2 N! \prod_{j=1}^{N} \sum_{\substack{m_j=1 \\ m_j \neq t_1,\dots,t_n, s_1,\dots,s_n}}^{M} \langle \Omega_0| \phi_{m_1} \cdots \phi_{m_N}\phi^+_{m_1} \cdots \phi^+_{m_N} |\Omega_0\rangle$$
$$\times \psi_{n+N,L}(\varepsilon t_1 \cdots \varepsilon t_n, \varepsilon m_1 \cdots \varepsilon m_N \mid \vartheta_0, \vartheta_L)$$
$$\times \psi_{n+N,L}(\varepsilon t_1 \cdots \varepsilon t_n, \varepsilon m_1 \cdots \varepsilon m_N \mid \vartheta_0, \vartheta_L)$$
$$= \prod_{i=1}^{N+n} \sum_{m_i=1}^{M} \prod_{j=1}^{N+n} \sum_{l_j=1}^{M} \langle \Omega_0| \phi_{m_1} \cdots \phi_{m_{n+N}}\phi_{s_1} \cdots \phi_{s_n}\phi_{t_1} \cdots \phi_{t_n}\phi_{l_1} \cdots \phi_{l_{1+N}} |\Omega_0\rangle$$
$$\times \psi_{n+N,L}(\varepsilon m_1 \cdots \varepsilon m_{n+N} \mid \vartheta_0, \vartheta_L)\, \psi_{n+N,L}(\varepsilon l_1 \cdots \varepsilon l_{n+N} \mid \vartheta_0, \vartheta_L) \tag{2.18}$$
$$\langle \Omega_0| \phi_{m_1} \cdots \phi_{m_N}\phi^+_{m_1} \cdots \phi^+_{m_N} |\Omega_0\rangle = 1 \tag{2.19}$$

This formula (2.17) is our standing point. The case $\vartheta_0, \vartheta_L = 0$ corresponds to Neumann boundary condition and the case $\vartheta_0, \vartheta_L = \infty$ to Dirichlet boundary condition. In the sequal, we use the following abbreviations.

$$\rho_{n,N,L}(x_1,\dots, x_n \mid x'_1 \dots, x'_n \mid +) = \rho_{n,N,L}(x_1,\dots, x_n \mid x'_1,\dots, x'_n \mid 0, 0) \tag{2.20}$$
$$\rho_{n,N,L}(x_1,\dots, x_n \mid x'_1,\dots, x'_n \mid -) = \rho_{n,N,L}(x_1,\dots, x_n \mid x'_1,\dots, x'_n \mid \infty, \infty) \tag{2.21}$$

In the sequel, for simplicity, we consider two important case: Neumann boundary conditions and Dirichlet boundary conditions.

Remark. There exists the simple relations between Neumann or Dirichlet boundaries and periodic boundaries. We can embed the differential equations for n point correlation functions of Neumann or Dirichlet boundaries, to the one for $2n$ or $2n-1$ point correlation functions without boundaries.

In Section 3, we derive the following formula.

Theorem 2.1. The correlation functions for an impenetrable bose gas with Neumann or Dirichlet boundaries are given by the following formulas.

$$\begin{aligned} &\rho_{n,N,L}(x'_1,..., x'_n \mid x''_1,..., x''_n \mid \varepsilon) \\ &= \left(-\frac{1}{2}\right)^n \prod_{1 \leqslant j<k \leqslant n} \operatorname{sgn}(x'_k - x'_j) \operatorname{sgn}(x''_k - x''_j) \\ &\quad \times \det\left(1-\frac{2}{\pi}\hat{K}_{\varepsilon,N,I_p} \,\middle|\, \begin{matrix} x'_1, x'_2,..., x'_n \\ x''_1, x''_2,..., x''_n \end{matrix}\right) \end{aligned} \tag{2.22}$$

where $\varepsilon = \pm$ and $0 \leqslant x'_j, x''_j \leqslant L$, $(j=1,..., n)$. Here I_p is the union of n intervals $I_p=[x_1, x_2] \cup \cdots \cup [x_{2n-1}, x_{2n}]$, where $0 \leqslant x_1 \leqslant \cdots \leqslant x_{2n} \leqslant L$ is the re-ordering of $x'_1,..., x'_n, x'_n, x''_1,..., x''_n$. The symbol

$$\det\left(1-\lambda\hat{K}_{\varepsilon,N,I_p} \,\middle|\, \begin{matrix} x'_1, x'_2,..., x'_n \\ x''_1, x''_2,..., x''_n \end{matrix}\right)$$

denotes the nth Fredholm minor corresponding to the following Fredholm type integral equation of the second kind.

$$((1-\lambda\hat{K}_{\varepsilon,N,I_p}) f)(x) = g(x), \qquad (x \in I_p) \tag{2.23}$$

Here the integral operator $\hat{K}_{\varepsilon,N,I_p}$ is defined by

$$\begin{aligned} &(\hat{K}_{\varepsilon,N,I_p} f)(x) \\ &= \int_{I_p} \left\{ \frac{\sin \dfrac{2(n+N)+1}{2L} \pi(x-y)}{\sin \dfrac{1}{2L} \pi(x-y)} + \varepsilon \frac{\sin \dfrac{2(n+N)+1}{2L} \pi(x+y)}{\sin \dfrac{1}{2L} \pi(x+y)} \right\} f(y)\, dy \end{aligned} \tag{2.24}$$

Using the above Fredholm minor formulas, we can take the thermodynamic limit for correlation functions, i.e., N, $L \to \infty$, $N/L = \rho_0$: fixed.

Corollary 2.2. The correlation functions for an impenetrable bose gas with Neumann or Dirichlet boundaries are given by the following formulas in the thermodynamic limit.

$$\begin{aligned}\rho_n(x'_1,..., x'_n \,|\, x''_1,..., x''_n \,|\, \varepsilon) &= \lim_{N, L \to \infty, N/L = \rho_0} \rho_{n,N,L}(x'_1,..., x'_n \,|\, x''_1,..., x''_n \,|\, \varepsilon) \\ &= \left(-\frac{1}{2}\right)^n \prod_{1 \leqslant j < k \leqslant n} \operatorname{sgn}(x'_k - x'_j) \operatorname{sgn}(x''_k - x''_j) \\ &\quad \times \det\left(1 - \frac{2}{\pi} \hat{K}_{\varepsilon, I_p} \,\middle|\, \begin{matrix} x'_1, x'_2 ..., x'_n \\ x''_1, x''_2,..., x''_n \end{matrix}\right) \end{aligned} \tag{2.25}$$

where $0 \leqslant x'_j$, $x''_j < +\infty$, $(j = 1,..., n)$. The symbol

$$\det\left(1 - \lambda \hat{K}_{\varepsilon, I_p} \,\middle|\, \begin{matrix} x'_1, x'_2,..., x'_n \\ x''_1, x''_2,..., x''_n \end{matrix}\right)$$

represents the nth Fredholm minor corresponding to the following Fredholm type integral equation of the second kind,

$$((1 - \lambda \hat{K}_{\varepsilon, I_p}) f)(x) = g(x), \qquad (x \in I_p) \tag{2.26}$$

where the integral operator $\hat{K}_{\varepsilon, I_p}$ is defined by

$$(\hat{K}_{\varepsilon, I_p} f)(x) = \int_{I_p} \left\{ \frac{\sin \rho_0 \pi (x - y)}{x - y} + \varepsilon \frac{\sin \rho_0 \pi (x + y)}{x + y} \right\} f(y)\, dy \tag{2.27}$$

In the sequel, we choose such a scale that $\pi \rho_0 = 1$.

In Section 4, we derive the differential equations for the correlation functions. Jimbo, Miwa, Môri, and Sato[8] introduction the generalization of the fifth Painlevé equations, hereafter refered to as the JMMS equations. Their simplest case is exactly the fifth Painlevé equation. We reduce the differential equations for Neumann or Dirichlet boundary case to that for without-boundary case, using the reflection relation in Lemma 4.2. For $n = 1$ and Dirichlet boundary case:

$$\rho_1(0 \,|x|\, -) = 0 \tag{2.28}$$

Next we explain $n = 1$ and Neumann boundary case. The differential equation for $\rho_1(0 \,|x|\, +)$ is described by the solutions of the Hamiltonian equations which was introduced in ref. 8 as the special case of the generalization of the fifth Painlevé equations. We cannot describe the correlation function $\rho_1(0\ |x|+)$ in terms of the fifth Painlevé ordinary differential equation. We need the many variable case of the JMMS equations.

$$\frac{d}{dx}\log\rho_1(0\ |x|+) = H_2(-x, 0, x) \tag{2.29}$$

Here $H_2(a_0, a_1, a_2)$ is the coefficient of the following Hamiltonian

$$\begin{aligned} H &= H_0(a_0, a_1, a_2)\, da_0 + H_1(a_0, a_1, a_2)\, da_1 + H_2(a_0, a_1, a_2)\, da_2 \\ &= -\sum_{j=0,2} \tfrac{1}{2}(r_{+j}r_{-1} - r_{+1}r_{-j})(\tilde{r}_{+j}r_{-1} - r_{+1}\tilde{r}_{-j})\, d\log(a_j - a_1) \\ &\quad -(r_{+0}\tilde{r}_{-2} - \tilde{r}_{+2}r_{-0})(\tilde{r}_{+0}r_{-2} - r_{+2}\tilde{r}_{-0})\, d\log(a_0 - a_2) \\ &\quad + ir_{+1}r_{-1}\, da_1 + i\sum_{j=0,2}(r_{+j}\tilde{r}_{-j} - \tilde{r}_j r_{-j})\, da_j - d\log(a_0 - a_2) \end{aligned} \tag{2.30}$$

Here the functions $r_{\pm j} = r_{\pm j}(a_0, a_1, a_2)$, $(j = 0, 1, 2)$, $\tilde{r}_{\pm 0} = \tilde{r}_{\pm 0}(a_0, a_1, a_2)$, $\tilde{r}_{\pm 2} = \tilde{r}_{\pm 2}(a_0, a_1, a_2)$ satisfy the Hamiltonian equations

$$dr_{\pm j} = \{r_{\pm j}, H\}, \quad (j = 0, 1, 2), \qquad d\tilde{r}_{\pm 0} = \{\tilde{r}_{\pm 0}, H\}, \qquad d\tilde{r}_{\pm 2} = \{\tilde{r}_{\pm 2}, H\} \tag{2.31}$$

where the Poisson bracket is defined by

$$\{r_{+1}, r_{-1}\} = 1, \ \ \{r_{+0}, \tilde{r}_{-0}\} = \{\tilde{r}_{+0}, r_{-0}\} = 1, \ \ \{r_{+2}, \tilde{r}_{-2}\} = \{\tilde{r}_{+2}, r_{-2}\} = 1 \tag{2.32}$$

This Hamiltonian H depends on odd number variables a_0, a_1, a_2. In the case without boundary, the differential equations for the correlation functions are described by the Hamiltonian equations which depend on even number of variables.[8] Therefore this point is new for Neumann boundary case.

In the general case, we can embed the differential equations for n point correlation functions of Neumann or Dirichlet boundaries, to the one for $2n$ or $2n-1$ point correlation functions without boundaries.

Theorem 2.3. In the thermodynamic limit, the differential equation for the correlation functions becomes the following.

$$\begin{aligned} &d\log\rho_n(x'_1,..., x'_n\,|\,x''_1,..., x''_n\,|\,\varepsilon) \\ &\quad = (1-n)\,\omega_{\varepsilon, I_p}\left(\frac{2}{\pi}\right) + \sum_{j=1}^{n}\sum_{\sigma\in S_n}\omega_{\varepsilon, I_p}^{(x'_j, x''_{\sigma(j)})}\left(\frac{2}{\pi}\right) \end{aligned} \tag{2.33}$$

where we denote by d the exterior differentiation with respect to $x'_1,\ldots, x'_n$, $x''_1,\ldots, x''_n$. Here the differential forms $\omega_{\varepsilon, I_p}(\lambda)$ and $\omega^{(y, y')}_{\varepsilon, I_p}(\lambda)$ are defined in Proposition 4.3 and Proposition 4.4, respectively. The differential forms $\omega_{\varepsilon, I_p}(\lambda)$ and $\omega^{(x'_j, x''_k)}_{\varepsilon, I_p}(\lambda)$ are described in terms of solutions of the generalized fifth Painlevé equations which were introduced by Jimbo, Miwa, Môri, and Sato.[8] Both Neumann and Dirichlet boundary conditions, $\omega_{\varepsilon, I_p}(\lambda)$ are described by the same solutions of the same differential equations.

Physically, it is interesting to derive the long distance asymptotics of correlation functions:

$$\rho_n(x'_1,\ldots, x'_n \,|\, x''_1,\ldots, x''_n \,|\, \varepsilon) \tag{2.34}$$

From the above theorem, we can reduce the evaluation of the asymptotics to the following two step problem.

1. Evaluate the asymptotics of the solution of the generalized fifth Painlevé introduced in ref. 8. (For our purpose, we only have to consider the special solution related to the correlation functions for the impenetrable Bose gas without boundary.)
2. Determine the asymptotic solutions of the differential Eq. (2.33) under the appropriate initial condition. (The main point is to determine the constant multiple in the asymptotics.)

In the case reducible to an ordinary differential equation, the above two problems have been already solved. Jimbo, Miwa, Môri, and Sato[8] considered the problem 1 of the correlation functions for the impenetrable Bose gas without boundary. McCoy and Tang[10] generalized the asymptotic formulas[8] to the 2-parameter solution of Painlevé V, which is analytic at the origin. Vaidya and Tracy[11, 12] considered the problem 2 of two-point correlation functions for the impenetrable Bose gas without boundary. (The pioneering work for Ising model was done by McCoy, Tracy, and Wu.[13]) In our case, to evaluate the asymptotics of $\rho_1(0\,|x|+)$, we have to consider the case of three-variables. However the asymptotics in many variable case is a non-trivial open problem. Therefore the above two problems for many variables case are our future problems.

3. FREDHOLM MINOR DETERMINANT FORMULAS

The purpose of this section is to give a proof of Theorem 2.1. Set $V = \mathbf{C}^2$. For $\varepsilon = \pm$, define operators $\eta^+(\theta, \varepsilon)$, $\eta(\theta, \varepsilon)$ acting on $V^{\otimes M}$ by

$$\eta^{+}(\theta, \varepsilon) = \sum_{m=1}^{M} (e^{-im\theta} + \varepsilon e^{im\theta}) \psi_m^{+} \tag{3.1}$$

$$\eta(\theta, \varepsilon) = \sum_{m=1}^{M} (e^{im\theta} + \varepsilon e^{-im\theta}) \psi_m \tag{3.2}$$

In the sequel we use the notation $\theta_{\mu,M} = \mu/(M+1)\pi$. The operators $\eta^{+}(\theta_{\mu,M}, \varepsilon)$, $\eta(\theta_{\mu,M}, \varepsilon)$ have the following anti-commutation relations for $\varepsilon = \pm$, $-M \leqslant \mu \leqslant M$.

$$\{\eta^{+}(\theta_{\mu,M}, \varepsilon), \eta^{+}(\theta_{\nu,M}, \varepsilon)\} = \{\eta(\theta_{\mu,M}, \varepsilon), \eta(\theta_{\nu,M}, \varepsilon)\} = 0 \tag{3.3}$$

$$\{\eta^{+}(\theta_{\mu,M}, \varepsilon), \eta(\theta_{\nu,M}, \varepsilon)\} = 2(M+1)(\delta_{\mu,\nu} + \varepsilon\delta_{\mu,-\nu}) \tag{3.4}$$

In the sequel we use the following abbreviations.

$$|\Omega_{N,M}(+)\rangle = |\Omega_{N,M}(0,0)\rangle, \qquad |\Omega_{N,M}(-)\rangle = |\Omega_{N,M}(\infty, \infty)\rangle \tag{3.5}$$

$$\langle\Omega_{N,M}(+)| = \langle\Omega_{N,M}(0,0)|, \qquad \langle\Omega_{N,M}(-)| = \langle\Omega_{N,M}(\infty, \infty)| \tag{3.6}$$

Using the operators $\eta^{+}(\theta, \varepsilon)$, $\eta(\theta, \varepsilon)$, we can write

$$|\Omega_{N,M}(\varepsilon)\rangle = \sqrt{\frac{1}{(2\varepsilon L)^N}}\, \eta^{+}(\theta_{1,M}, \varepsilon)\, \eta^{+}(\theta_{2,M}, \varepsilon) \cdots \eta^{+}(\theta_{N,M}, \varepsilon)\, |\Omega_0\rangle \tag{3.7}$$

$$\langle\Omega_{N,M}(\varepsilon)| = \varepsilon^N \sqrt{\frac{1}{(2\varepsilon L)^N}}\, \langle\Omega_0|\, \eta(\theta_{N,M}, \varepsilon) \cdots \eta(\theta_{2,M}, \varepsilon)\, \eta(\theta_{1,M}, \varepsilon) \tag{3.8}$$

The operators $\eta^{+}(\theta_{\mu,M}, \varepsilon)$, $\eta(\theta_{\mu,M}, \varepsilon)$ act on the vectors $|\Omega_{N,M}(\varepsilon)\rangle$, $\langle\Omega_{N,M}(\varepsilon)|$ as follows.

$$\text{For} \quad |\mu| \leqslant N, \quad \eta^{+}(\theta_{\mu,M}, \varepsilon)\, |\Omega_{N,M}(\varepsilon)\rangle = 0, \quad \langle\Omega_{N,M}(\varepsilon)|\, \eta(\theta_{\mu,M}, \varepsilon) = 0 \tag{3.9}$$

$$\text{For} \quad |\mu| > N, \quad \langle\Omega_{N,M}(\varepsilon)|\, \eta^{+}(\theta_{\mu,M}, \varepsilon) = 0, \quad \eta(\theta_{\mu,M}, \varepsilon)\, |\Omega_{N,M}(\varepsilon)\} = 0 \tag{3.10}$$

Define operators p_m, q_m $(m = 1,..., M)$ by

$$p_m = \psi_m^{+} + \psi_m, \qquad q_m = -\psi_m^{+} + \psi_m \tag{3.11}$$

Set

$$p(\theta)=\sum_{m=1}^{M} p_m e^{-im\theta}, \qquad q(\theta)=\sum_{m=1}^{M} q_m e^{-im\theta} \tag{3.12}$$

We have

$$p(\theta)+\varepsilon p(-\theta)=\eta^+(\theta,\varepsilon)-\eta(\theta,\varepsilon) \tag{3.13}$$

$$q(\theta)+\varepsilon q(-\theta)=\eta^+(\theta,\varepsilon)+\eta(\theta,\varepsilon) \tag{3.14}$$

The following relations hold for $m\geqslant 1$.

$$\sum_{\mu=-M}^{M+1} p(\theta_{-\mu,M})\, e^{im\theta_{\mu,M}}=0, \qquad \sum_{\mu=-M}^{M+1} q(\theta_{-\mu,M})\, e^{im\theta_{\mu,M}}=0 \tag{3.15}$$

Therefore, p_m and q_m can be obtained by Fourier transformations.

$$p_m=\frac{1}{2(M+1)}\sum_{\mu=-M}^{M+1} (\eta^+(\theta_{\mu,M},\varepsilon)-\eta(\theta_{\mu,M},\varepsilon))\, e^{im\theta_{\mu,M}} \tag{3.16}$$

$$q_m=\frac{1}{2(M+1)}\sum_{\mu=-M}^{M+1} (\eta^+(\theta_{\mu,M},\varepsilon)+\eta(\theta_{\mu,M},\varepsilon))\, e^{im\theta_{\mu,M}} \tag{3.17}$$

We define

$$\langle \wp \rangle_{\varepsilon,N}=\frac{\langle \Omega_{N,M}(\varepsilon)|\ \wp\ |\Omega_{N,M}(\varepsilon)\rangle}{\langle \Omega_{N,M}(\varepsilon)\mid \Omega_{N,M}(\varepsilon)\rangle} \tag{3.18}$$

for $\wp\in \mathrm{End}((\mathbf{C}^2)^{\otimes M})$. We call $\langle \wp \rangle_{\varepsilon,N}$ the expectation value of the operator $\wp$.

Lemma 3.1. The expectation values of the two products of $\eta^+(\theta_{\mu,M},\varepsilon)$ and $\eta(\theta_{\mu,M},\varepsilon)$ are given by

$$\begin{pmatrix} \langle \eta^+(\theta_{\mu,M},\varepsilon)\,\eta^+(\theta_{\nu,M})\rangle_{\varepsilon,N} & \langle \eta^+(\theta_{\mu,M},\varepsilon)\,\eta(\theta_{\nu,M},\varepsilon)\rangle_{\varepsilon,N} \\ \langle \eta(\theta_{\mu,M},\varepsilon)\,\eta^+(\theta_{\nu,M},\varepsilon)\rangle_{\varepsilon,N} & \langle \eta(\theta_{\mu,M},\varepsilon)\,\eta(\theta_{\nu,M},\varepsilon)\rangle_{\varepsilon,N} \end{pmatrix}$$
$$=2(M+1)(\delta_{\mu,\nu}+\varepsilon\delta_{\mu,-\nu})\begin{pmatrix} 0 & \theta_1(N-|\mu|) \\ \theta_2(|\mu|-N) & 0 \end{pmatrix} \tag{3.19}$$

where $-M\leqslant\mu,\ \nu\leqslant M$ and $\theta_1(x)$, $\theta_2(x)$ are the step function

$$\theta_1(x)=\begin{cases}1, & x\geqslant 0\\ 0, & x<0\end{cases} \qquad \theta_2(x)=\begin{cases}1, & x>0\\ 0, & x\leqslant 0\end{cases} \tag{3.20}$$

Proposition 3.2. The expectation values of the products of p_m, q_m are given by

$$\begin{pmatrix} \langle p_l p_m \rangle_{\varepsilon,N} & \langle p_l q_m \rangle_{\varepsilon,N} \\ \langle q_l p_m \rangle_{\varepsilon,N} & \langle q_l q_m \rangle_{\varepsilon,N} \end{pmatrix} = \begin{pmatrix} \delta_{l,m} & -K_{\varepsilon,l,m} \\ K_{\varepsilon,l,m} & \delta_{l,m} \end{pmatrix} \tag{3.21}$$

where $l, m = 1, 2,..., M$ and

$$K_{\varepsilon,l,m} = \delta_{l,m} - \frac{1}{M+1} \left\{ \frac{\sin \frac{2N+1}{2(M+1)}(l-m)\pi}{\sin \frac{1}{2(M+1})(l-m)\pi} + \varepsilon \frac{\sin \frac{2N+1}{2(M+1)}(l+m)\pi}{\sin \frac{1}{2(M+1)}(l+m)\pi} \right\} \tag{3.22}$$

Proof. By direct calculation, we can check the following.

$$\begin{aligned} &\langle p_l q_m \rangle_{\varepsilon,N} \\ &= \frac{1}{(2(M+1))^2} \sum_{\mu=-M}^{M+1} \sum_{\nu=-M}^{M+1} \langle (\eta^+ - \eta)(\theta_{\mu,M}, \varepsilon) \\ &\quad \times (\eta^+ + \eta)(\theta_{\nu,M}, \varepsilon) \rangle_{\varepsilon,N} \, e^{il\theta_{\mu,M}} e^{im\theta_{\nu,M}} \\ &= \frac{1}{2(M+1)} \sum_{\mu=-M}^{M+1} \sum_{\nu=-M}^{M+1} (\delta_{\mu,\nu} + \varepsilon\delta_{\mu,-\nu}) \, \varepsilon_+(N-|\nu|) \, e^{i(l\theta_{\mu,M} + m\theta_{\nu,M})} \\ &= -\delta_{l,m} + \frac{1}{M+1} \left\{ \frac{\sin \frac{2N+1}{2(M+1)}(l-m)\pi}{\sin \frac{1}{2(M+1)}(l-m)\pi} + \varepsilon \frac{\sin \frac{2N+1}{2(M+1)}(l+m)\pi}{\sin \frac{1}{2(M+1)}(l+m)\pi} \right\} \end{aligned} \tag{3.23}$$

Here $\varepsilon_+(x)$ denotes the sign function

$$\varepsilon_+(x) = \begin{cases} 1, & x \geqslant 0 \\ -1 & x < 0 \end{cases}$$

We have used the relation,

$$\sum_{\mu=-M}^{M+1} \varepsilon_+(N-|\mu|) \, ^{is\theta_\mu} = 2 \left\{ (M+1)\,\delta_{s,0} - \frac{\sin \frac{2N+1}{2(M+1)}\pi s}{\sin \frac{1}{2(M+1)}\pi_s} \right\} \quad \blacksquare \tag{3.24}$$

We prepare some notations. Choose $0 \leqslant m_1 < \cdots < m_n \leqslant M$ and $0 \leqslant m_{n+1} < \cdots < m_{2n} \leqslant M$. Let $m'_1 \leqslant m'_2 \leqslant \cdots \leqslant m'_{2n}$ such that $m'_j = m_{\sigma(j)}$ $(\sigma \in S_{2n})$. Define the interval $I_{j,M}$ and I_M by $I_{j,M} = \{l \in \mathbf{Z} \mid m'_{2j-1} + 1 \leqslant l \leqslant m'_{2j}\}$, $I_M = I_{1,M} \cup I_{2,M} \cup \cdots \cup I_{n,M}$. Define t_m, $t_{I_M} \in \mathrm{End}((\mathbf{C}^2)^{\otimes M})$ by $t_m = q_1 p_1 \cdots q_m p_m$, $t_{I_M} = t_{m'_1} \cdots t_{m'_{2n}}$. Define $R_{\varepsilon, ppI_M}(l, m)$, $R_{\varepsilon, pqI_M}(l, m)$, $R_{\varepsilon, qpI_M}(l, m)$ and $R_{\varepsilon, qqI_M}(l, m)$ by

$$\begin{pmatrix} R_{\varepsilon, ppI_M}(l, m) & R_{\varepsilon, pqI_M}(l, m) \\ R_{\varepsilon, qpI_M}(l, m & R_{\varepsilon, qqI_M}(l, m) \end{pmatrix} = \frac{1}{\langle t_{I_M} \rangle_{\varepsilon, N}} \begin{pmatrix} \langle p_l p_m t_{I_M} \rangle_{\varepsilon, N} & \langle p_l q_m t_{I_M} \rangle_{\varepsilon, N} \\ \langle q_l p_m t_{I_M} \rangle_{\varepsilon, N} & \langle q_l q_m t_{I_M} \rangle_{\varepsilon, N} \end{pmatrix} \tag{3.25}$$

where $l, m = 1, 2, \ldots, M$. Define the matrix K_{ε, I_M} by $(K_{\varepsilon, I_M})_{j,k \in I_M} = \langle q_j p_k \rangle_{\varepsilon, N}$.

Lemma 3.3. The expectation value of t_{I_M} is given by

$$\langle t_{I_M} \rangle_{\varepsilon, N} = \det K_{\varepsilon, I_m} \tag{3.26}$$

For $l, m = 1, \ldots, M$, the following relation holds.

$$R_{\varepsilon, ppI_M}(l, m) + R_{\varepsilon, qqI_M}(l, m) = 0, \qquad R_{\varepsilon, pqI_M}(l, m) + R_{\varepsilon, qpI_M}(l, m) = 0 \tag{3.27}$$

Furthermore $R_{\varepsilon, ppI_M}(l, m)$, $R_{\varepsilon, pqI_M}(l, m)$, $R_{\varepsilon, qpI_M}(l, m)$ and $R_{\varepsilon, qqI_M}(l, m)$ have simple formulas. For $l, m \in I_M$, the following relations hold.

$$\begin{pmatrix} R_{\varepsilon, ppI_M}(l, m) & R_{\varepsilon, pqI_M}(l, m) \\ R_{\varepsilon, qpI_M}(l, m) & R_{\varepsilon, qqI_M}(l, m) \end{pmatrix} = \begin{pmatrix} \delta_{l,m} & -(K_{\varepsilon, I_M}^{-1})_{l,m} \\ (K_{\varepsilon, I_M}^{-1})_{l,m} & -\delta_{l,m} \end{pmatrix} \tag{3.28}$$

Proof. From Wick's theorem and $\langle p_l p_m \rangle_{\varepsilon, N} = \delta_{l,m}$, we obtain $\langle p_l p_m t_{I_M} \rangle_{\varepsilon, N} = \langle t_{I_M} \rangle_{\varepsilon, N} \delta_{l,m}$. From this and Wick's theorem, we can deduce

$$\begin{aligned} \langle t_{I_M} \rangle_{\varepsilon, N} \delta_{m,m'} &= \langle p_{m'} p_m t_{I_M} \rangle_{\varepsilon, N} \\ &= \langle p_{m'} p_m \rangle_{\varepsilon, N} \langle t_{I_M} \rangle_{\varepsilon, N} + \sum_{\lambda \in I_M} \langle p_{m''} q_\lambda \rangle_{\varepsilon, N} \langle p_m q_\lambda t_{I_M} \rangle_{\varepsilon, N} \\ &\quad - \sum_{\lambda \in I_M} \langle p_{m'} p_\lambda \rangle_{\varepsilon, N} \langle p_m p_\lambda t_{I_M} \rangle_{\varepsilon, N} \\ &= \sum_{\lambda \in I_M} \langle p_{m'} q_\lambda \rangle_{\varepsilon, N} \langle p_m q_\lambda t_{I_M} \rangle_{\varepsilon, N} \\ &= -\langle t_{I_M} \rangle_{\varepsilon, N} \sum_{\lambda \in I_M} R_{\varepsilon, pqI_M}(m, \lambda) (K_{\varepsilon, I_M})_{\lambda, m'} \quad \blacksquare \end{aligned} \tag{3.29}$$

We prepare some notations. Set

$$K_{\varepsilon,N}(x,x')=\frac{\pi}{2L}\left\{\frac{\sin\frac{2(n+N)+1}{2L}\pi(x-x')}{\sin\frac{1}{2L}\pi(x-x')}+\varepsilon\frac{\sin\frac{2(n+N)+1}{2L}\pi(x+x')}{\sin\frac{1}{2L}\pi(x+x')}\right\} \tag{3.30}$$

Define the integral operator $\hat{K}_{\varepsilon,N,J}$ by

$$(\hat{K}_{\varepsilon,N,J}f)(x)=\int_J K_{\varepsilon,N}(x,y)\,f(y)\,dy \tag{3.31}$$

Let us denote by $\det(1-\lambda\hat{K}_{\varepsilon,N,J})$ and

$$\det\left(1-\lambda\hat{K}_{\varepsilon,N,J}\,\middle|\,\begin{matrix}x_1,\ldots,x_n\\x'_1,\ldots,x'_n\end{matrix}\right)$$

the Fredholm determinant and the nth Fredholm minor determinant, respectively. Namely, we have

$$\det(1-\lambda\hat{K}_{\varepsilon,N,J})=\sum_{l=0}^{\infty}\frac{(-\lambda)^l}{l!}\int_J\cdots\int_J dx_1\cdots dx_l\,K_{\varepsilon,N}\begin{pmatrix}x_1,\ldots,x_l\\x_1,\ldots,x_l\end{pmatrix} \tag{3.32}$$

$$\det\left(1-\lambda\hat{K}_{\varepsilon,N,J}\,\middle|\,\begin{matrix}x_1,\ldots,x_n\\x'_1,\ldots,x'_n\end{matrix}\right)=\sum_{l=0}^{\infty}\frac{(-\lambda)^{l+n}}{l!}\int_J\cdots\int_J dx_{n+1}\cdots dx_{n+l}$$
$$\times K_{\varepsilon,N}\begin{pmatrix}x_1,\ldots,x_n & x_{n+1},\ldots,x_{n+l}\\x'_1,\ldots,x'_n & x_{n+1},\ldots,x_{n+l}\end{pmatrix} \tag{3.33}$$

where we have used

$$K_{\varepsilon,N}\begin{pmatrix}x_1,\ldots,x_l\\x'_1,\ldots,x'_l\end{pmatrix}=\det_{1\leqslant j,k\leqslant l}(K_{\varepsilon,N}(x_j,x'_k)) \tag{3.34}$$

Set

$$R_{\varepsilon,N,J}(x,x'\mid\lambda)$$
$$=\sum_{l=0}^{\infty}\lambda^l\int_J\cdots\int_J dx_1\cdots dx_l\,K_{\varepsilon,N}(x,x_1)\,K_{\varepsilon,N}(x_1,x_2)\cdots K_{\varepsilon,N}(x_l,x') \tag{3.35}$$

Define the integral operator $\hat{R}_{\varepsilon,N,J}$ by

$$(\hat{R}_{\varepsilon,N,J}f)(x)=\int_J R_{\varepsilon,N,J}(x,y\,|\,\lambda)\,f(y)\,dy \tag{3.36}$$

The resolvent kernel $R_{\varepsilon,N,J}(x,x'\,|\,\lambda)$ can be characterized by the following integral equation

$$(1-\lambda\hat{K}_{\varepsilon,N,J})(1+\lambda\hat{R}_{\varepsilon,N,J})=1 \tag{3.37}$$

Here we present a proof of Theorem 2.1.

Proof of Theorem 2.1. First, for simplicity, we show the $n=1$ case. For $s_1\leqslant s_2$, $(s_1,s_2\in\{1,2,\ldots,M\})$, we have

$$\begin{aligned}\langle\phi_{s_1}\phi_{s_2}^+\rangle_{\varepsilon,N}&=\tfrac{1}{2}\langle(\phi_{s_1}^+ +\phi_{s_1})(\phi_{s_2}^+ +\phi_{s_2})\rangle_{\varepsilon,N}\\&=\tfrac{1}{2}\langle(\psi_{s_1}^+ -\psi_{s_1})\,\sigma_{s_1+1}^z\cdots\sigma_{s_2-1}^z(\psi_{s_2}^+ +\psi_{s_2})\rangle_{\varepsilon,N}\\&=\tfrac{1}{2}(-1)^{s_2-s_1}\langle(q_{s_1}p_{s_1+1})(q_{s_1+1}p_{s_1+2})\cdots(q_{s_2-1}p_{s_2})\rangle_{\varepsilon,N}\end{aligned} \tag{3.38}$$

Applying Wick's theorem and $\langle p_jp_k\rangle_{\varepsilon,N}=\delta_{j,k}$ and $\langle q_jq_k\rangle_{\varepsilon,N}=\delta_{j,k}$, we can write the above as a determinant

$$\langle\phi_{s_1}\phi_{s_2}^+\rangle_{\varepsilon,N}=\tfrac{1}{2}\det_{s_1\leqslant j,k\leqslant s_2-1}(\langle p_{j+1}q_k\rangle_{\varepsilon,N}) \tag{3.39}$$

From (2.17), $\rho_{1,N,L}(x_1\,|x_1'|\,\varepsilon)=\lim_{\varepsilon\to 0}\varepsilon^N\langle\phi_{s_1}\phi_{s_2}^+\rangle_{\varepsilon,1+N}$ holds, where $\varepsilon=L/(M+1)$ and $\varepsilon s_1\to x_1$, $\varepsilon s_2\to x_1'$. Set $v=s_2-s_1$. From Proposition 3.2, we obtain

$$\begin{aligned}&\rho_{1,N,L}(x_1\,|x_1'|\,\varepsilon)\\&\quad=\frac{-1}{2(x_1'-x_1)}\lim_{v\to\infty}v\det_{1\leqslant j,k\leqslant v}\left(-\delta_{j+1,k}+\frac{1}{v}G_\varepsilon\left(\frac{j+1+s_1}{v},\frac{k}{v}\right)\right)\end{aligned} \tag{3.40}$$

where we set

$$\begin{aligned}G_\varepsilon(y,y')=\frac{x_1'-x_1}{L}\Bigg\{&\frac{\sin\dfrac{2(1+N)+1}{2L}\pi(y+y')(x_1'-x_1)}{\sin\dfrac{1}{2L}\pi(y+y')(x_1'-x_1)}\\&+\varepsilon\frac{\sin\dfrac{2(1+N)+1}{2L}\pi(y-y')(x_1'-x_1)}{\sin\dfrac{1}{2L}\pi(y-y')(x_1'-x_1)}\Bigg\}\end{aligned} \tag{3.41}$$

We apply the following relation to the above equation

$$\lim_{\nu\to\infty} \nu \det_{1\leqslant j,k\leqslant \nu}\left(-\delta_{j+1,k}+\frac{1}{\nu}\lambda H\left(\frac{j+1}{\nu},\frac{k}{\nu}\right)\right)$$
$$= -(-\lambda)\,H(0,1)-(-\lambda)^2\int_0^1 dy_2\,H\begin{pmatrix}0 & y_2\\ 1 & y_2\end{pmatrix}-\cdots$$
$$-(-\lambda)^{m+1}\frac{1}{m!}\int_0^1\cdots\int_0^1 dy_2\cdots dy_{m+1}\,H\begin{pmatrix}0 & y_2,\ldots, y_{m+1}\\ 1 & y_2,\ldots, y_{m+1}\end{pmatrix}-\cdots \tag{3.42}$$

Here $H(y_1, y_2)$ is a continuous function, and we use

$$H\begin{pmatrix}y_1,\ldots, y_m\\ y'_1,\ldots, y'_m\end{pmatrix}=\det_{1\leqslant j,k\leqslant m}(H(y_j, y'_k)) \tag{3.43}$$

We can write down

$$\rho_{1,N,L}(x_1\,|x'_1|\,\varepsilon)$$
$$=\left(-\frac{1}{2}\right)\left[\left(-\frac{2}{\pi}\right)K_{\varepsilon,N}(x_1,x'_1)+\left(-\frac{2}{\pi}\right)^2\int_{x_1}^{x'_1}dy_2\,K_{\varepsilon,N}\begin{pmatrix}x_1 & y_2\\ x'_1 & y_2\end{pmatrix}+\cdots\right.$$
$$+\left(-\frac{2}{\pi}\right)^{m+1}\frac{1}{m!}\int_{x_1}^{x'_1}\cdots\int_{x_1}^{x'_1}dy_2\cdots dy_{m+1}\,K_{\varepsilon,N}$$
$$\left.\times\begin{pmatrix}x_1 & y_2,\ldots, y_{m+1}\\ x'_1 & y_2,\ldots, y_{m+1}\end{pmatrix}+\cdots\right] \tag{3.44}$$

Now, we have proved $n=1$ case. Next we shall prove the general case. From Proposition 3.2 and Lemma 3.3, we can deduce,

$$\lim_{M\to\infty}\langle t_{I_M}\rangle_{\varepsilon,n+N}=\det(1-\lambda\hat{K}_{\varepsilon,N,I_p})|_{\lambda=2/\pi} \tag{3.45}$$

From Lemma 3.3, we see $\sum_{l\in I_M}(K_{\varepsilon,I_M})_{m,l}R_{\varepsilon,qpI_M}(l,m')=\delta_{m,m'}$. Comparing this relation to the relation $(1-\lambda\hat{K}_{\varepsilon,N,I_p})(1+\lambda\hat{R}_{\varepsilon,N,I_p})=1$, we can deduce the following

$$\lim_{M\to\infty}\left(\frac{M}{L}\right)R_{\varepsilon,qpI_M}(m_j,m_k)=\lambda R_{\varepsilon,N,I_p}(x_j,x_k\,|\,\lambda)\Big|_{\lambda=2/\pi} \tag{3.46}$$

for $m_j \neq m_k$, $(L/M)\, m_j \to x_j$ $(j = 1,..., n)$. Choose $0 \leqslant m_1 < \cdots < m_n \leqslant M$ and $0 \leqslant m_{n+1} < \cdots < m_{2n} \leqslant M$. Let $m'_1 \leqslant m'_2 \leqslant \cdots \leqslant m'_{2n}$ such that $m'_j = m_{\sigma(j)}$ $(\sigma \in S_{2n})$. Set

$$m''_j = \begin{cases} m_j, & \sigma(j): \quad \text{odd} \\ m_j + 1, & \sigma(j): \quad \text{even} \end{cases}$$

From the parity argument, we obtain

$$\langle t_{m'_1} \cdots \phi_{m''_j} \cdots \phi_{m''_k} \cdots t_{m'_{2n}} \rangle_{\varepsilon, n+N} = 0, \qquad \langle t_{m'_1} \cdots \phi^+_{m''_j} \cdots \phi^+_{m''_k} \cdots t_{m'_{2n}} \rangle_{\varepsilon, n+N} = 0 \tag{3.47}$$

The expectation value $\langle \phi_{m''_1} \phi_{m''_2} \cdots \phi_{m''_n} \phi^+_{m''_{n+1}} \phi^+_{m''_{n+2}} \cdots \phi^+_{m''_{2n}} \rangle_{\varepsilon, n+N}$ can be written as Pfaffian. (See p. 967 of [?]). Furthermore, from (3.47), we can write the expectation value as a determinant

$$\begin{aligned} &\frac{\langle \phi_{m''_1} \phi_{m''_2} \cdots \phi_{m''_n} \phi^+_{m''_{n+1}} \phi^+_{m''_{n+2}} \cdots \phi^+_{m''_{2n}} \rangle_{\varepsilon, n+N}}{\langle t_{I_M} \rangle_{\varepsilon, n+N}} \\ &= (-1)^{(1/2)n(n-1)} \det_{1 \leqslant j, k \leqslant n} \left(\frac{\langle t_{m'_1} \cdots \phi_{m''_j} \cdots t_{m''_n} t_{m''_{n+1}} \cdots \phi^+_{m''_{n+k}} \cdots t_{m'_{2n}} \rangle_{\varepsilon, n+N}}{\langle t_{I_M} \rangle_{\varepsilon, n+N}} \right) \\ &= \left(-\frac{1}{2} \right)^n \det_{1 \leqslant j, k \leqslant n} (R_{\varepsilon, qpI_M}(m''_j, m''_{n+k})) \end{aligned} \tag{3.48}$$

From the Eqs. (3.45), (3.46), and (3.48), we can deduce

$$\begin{aligned} &\lim_{M \to \infty} \left(\frac{M}{L} \right)^n \langle \phi_{m''_1} \phi_{m''_2} \cdots \phi_{m''_n} \phi^+_{m''_{n+1}} \phi^+_{m''_{n+2}} \cdots \phi^+_{m''_{2n}} \rangle_{\varepsilon, n+N} \\ &\qquad = (-\lambda)^n \det(1 - \lambda \hat{K}_{\varepsilon, N, I_p}) \det_{1 \leqslant j, k \leqslant n} (R_{\varepsilon, N, I_p}(x_j, x'_k \mid \lambda))|_{\lambda = 2/\pi} \end{aligned} \tag{3.49}$$

where $(L/M)\, m'_j \to x_j$, $(L/M)\, m'_{n+j} \to x'_j$, $(j = 1,..., n)$, when $M \to \infty$. Using the Fredholm identity,

$$\begin{aligned} &(-\lambda)^n \det(1 - \lambda \hat{K}_{\varepsilon, N, I_p}) \det_{1 \leqslant j, k \leqslant n} (R_{\varepsilon, N, I_p}(x_j, x'_k \mid \lambda)) \\ &\qquad = \det \left(1 - \lambda \hat{K}_{\varepsilon, N, I_p} \middle| \begin{matrix} x_1, x_2 ..., x_n \\ x'_1, x'_2, ..., x'_n \end{matrix} \right) \end{aligned} \tag{3.50}$$

we can deduce the following

$$\lim_{M\to\infty}\left(\frac{M}{L}\right)^n \langle \phi_{m_1''}\phi_{m_2''}\cdots\phi_{m_n''}\phi^+_{m_{n+1}''}\phi^+_{m_{n+2}''}\cdots\phi^+_{m_{2n}''}\rangle_{\varepsilon,n+N}$$

$$=\left(-\frac{1}{2}\right)^n \det\left(1-\frac{2}{\pi}\hat{K}_{\varepsilon,N,I_p}\,\middle|\, \begin{matrix} x_1, x_2 ..., x_n \\ x_1', x_2',..., x_n'\end{matrix}\right) \tag{3.51}$$

This complete the proof of the general case. ∎

Fredholm minor series in this correlation function is a finite sum because

$$K_{\varepsilon,N}\begin{pmatrix} x_1,..., x_l \\ x_1',..., x_l'\end{pmatrix}=0 \qquad \text{for} \quad m\geqslant 2(n+N) \tag{3.52}$$

To see this, define an $m\times M$ matrix $A_M(\alpha\,|\,x_1,..., x_m)$ by

$$(A_M(\alpha\,|\,x_1,..., x_m))_{j,k}=e^{i\alpha k x_j}, \quad \text{for } j=1,..., m,\ k=-\tfrac{1}{2}(M-1),..., \tfrac{1}{2}(M-1) \tag{3.53}$$

Using this matrix, we obtain the following.

$$K_{\varepsilon,N}\begin{pmatrix} x_1,..., x_l \\ x_1',..., x_l'\end{pmatrix}$$

$$=\sum_{\varepsilon_1,...,\varepsilon_m=\pm}(\varepsilon_1\cdots\varepsilon_m)^{(1-\varepsilon)/2}\det_{1\leqslant j,k\leqslant m}\left(\frac{\pi}{2L}\frac{\sin\dfrac{2(n+N)+1}{2L}\pi(x_j-\varepsilon_k x_k')}{\sin\dfrac{1}{2L}\pi(x_j-\varepsilon_k x_k')}\right)$$

$$=\left(\frac{\pi}{2L}\right)^m\sum_{\varepsilon_1,...,\varepsilon_m=\pm}(\varepsilon_1\cdots\varepsilon_m)^{(1-\varepsilon)/2}\det\left(A_{2(n+N)+1}\left(\frac{\pi}{L}\,\middle|\,x_1,..., x_m\right)\right.$$

$$\left.\times A^T_{2(n+N)-1}\left(-\frac{\pi}{L}\,\middle|\,\varepsilon_1 x_1',..., \varepsilon_m x_m'\right)\right) \tag{3.54}$$

Here A^T represents the transposed matrix. ¿From elementary argument of linear algebra, we can see $\det(A_M(\alpha\,|\,x_1,..., x_m)\,A^T_M(\beta\,|\,x_1',..., x_m'))=0$, for $m\geqslant M+1$. Now we have proved (3.52).

4. GENERALIZED FIFTH PAINLEVÉ EQUATION

The purpose of this section is to give a proof of Theorem 2.3. Following ref. 8, we describe the correlation functions in terms of the generalization of the fifth Painlevé equations, which are given by Jimbo, Miwa, Môri, and Sato in the thermodynamic limit ($N, L \to \infty$, $N/L = \rho_0$: fixed). Set

$$K_\varepsilon(x, x') = \frac{\sin \rho_0 \pi (x - x')}{x - x'} + \varepsilon \frac{\sin \rho_0 \pi (x + x')}{x + x'} \tag{4.1}$$

Define the integral operators $\hat{K}_{\varepsilon, J}$ by

$$(\hat{K}_{\varepsilon, J} f)(x) = \int_J K_\varepsilon(x, y) f(y)\, dy \tag{4.2}$$

Set

$$R_{\varepsilon, J}(x, x' \mid \lambda) = \sum_{l=0}^{\infty} \lambda^l \int_J \cdots \int_J dx_1 \cdots dx_l\, K_\varepsilon(x, x_1)\, K_\varepsilon(x_1, x_2) \cdots K_\varepsilon(x_l, x') \tag{4.3}$$

Definne the integral operators $\hat{R}_{\varepsilon, J}$ by

$$(\hat{R}_{\varepsilon, J} f)(x) = \int_J R_{\varepsilon, J}(x, y \mid \lambda) f(y)\, dy \tag{4.4}$$

The resolvent kernel $R_{\varepsilon, J}(x, x' \mid \lambda)$ is characterized by the following integral equation,

$$(1 + \lambda \hat{R}_{\varepsilon, J})(1 + \lambda \hat{K}_{\varepsilon, J}) = 1 \tag{4.5}$$

Let us denote by $\det(1 - \lambda \hat{K}_{\varepsilon, J})$ and

$$\det\left(1 - \lambda \hat{K}_{\varepsilon, J} \,\middle|\, \begin{matrix} x_1, \ldots, x_n \\ x'_1, \ldots, x'_n \end{matrix}\right)$$

the Fredholm determinant and the nth Fredholm minor determinant, respectively. Namely, we set

$$\det(1 - \lambda \hat{K}_{\varepsilon, J}) = \sum_{l=0}^{\infty} \frac{(-\lambda)^l}{l!} \int_J \cdots \int_J dx_1 \cdots dx_l\, K_\varepsilon \begin{pmatrix} x_1, \ldots, x_l \\ x_1, \ldots, x_l \end{pmatrix} \tag{4.6}$$

$$\det\left(1 - \lambda \hat{K}_{\varepsilon, J} \,\middle|\, \begin{matrix} x_1, \ldots, x_n \\ x'_1, \ldots, x'_n \end{matrix}\right) = \sum_{l=0}^{\infty} \frac{(-\lambda)^{l+n}}{l!} \int_J \cdots \int_J dx_{n+1} \cdots dx_{n+l} \times K_\varepsilon \begin{pmatrix} x_1, \ldots, x_n & x_{n+1}, \ldots, x_{n+l} \\ x'_1, \ldots, x'_n & x_{n+1}, \ldots, x_{n+l} \end{pmatrix} \tag{4.7}$$

where we have used

$$K_\varepsilon\begin{pmatrix} x_1,..., x_l \\ x'_1,..., x'_l \end{pmatrix} = \det_{1 \leqslant j, k \leqslant l} (K_\varepsilon(x_j, x'_k)) \tag{4.8}$$

We set

$$\rho_n(x'_1,..., x'_n \mid x''_1,..., x''_n \mid \varepsilon) = \lim_{N, L \to \infty, N/L = \rho_0} \rho_{n,N,L}(x'_1,..., x'_n \mid x''_1,..., x''_n \mid \varepsilon) \tag{4.9}$$

Contrary to the case in a finite box, the Fredholm minor series in correlation functions is infinite series and correlation function $\rho_n(x'_1,..., x'_n \mid x''_1,..., x''_n \mid \varepsilon)$ becomes a transcendental function. In the sequel, we shall study the differential equations for correlation functions in the thermodynamic limit. In what follows we can choose such a scale that $\pi\rho_0 = 1$.

We prepare some notations. Let $-\infty < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{2m} < +\infty$. We denote by I the interval defined by $I = [a_1, a_2] \cup \cdots \cup [a_{2m-1}, a_{2m}]$. Set

$$L(x, x') = \frac{\sin(x - x')}{x - x'} \tag{4.10}$$

Define the integral operators $\hat{L}_I$ by

$$(\hat{L}_I f)(x) = \int_I L(x, y) f(y)\, dy \tag{4.11}$$

Set

$$S_I(x, x' \mid \lambda) = \sum_{l=0}^{\infty} \lambda^l \int_I \cdots \int_I dx_1 \cdots dx_l\, L(x, x_1)\, L(x_1, x_2) \cdots L(x_l, x') \tag{4.12}$$

Define the integral operator $\hat{S}_I$ by

$$(\hat{S}_I f)(x) = \int_I S_I(x, y \mid \lambda) f(y)\, dy \tag{4.13}$$

The resolvent kernel $S_I(x, x'\lambda)$ is characterized by the following integral equation,

$$(1 + \lambda\hat{S}_I)(1 - \lambda\hat{L}_I) = 1 \tag{4.14}$$

Set

$$S_I\begin{pmatrix} x_1,\dots,x_l \\ x'_1,\dots,x'_l \end{pmatrix}\lambda\Big) = \det_{1\leqslant j,k\leqslant l}(S_I(x_j, x'_k\,|\,\lambda)) \tag{4.15}$$

$$h_I(x) = \frac{1}{2\pi i}\log\left\{\frac{(x-a_1)(x-a_3)\cdots(x-a_{2n-1})}{(x-a_2)(x-a_4)\cdots(x-a_{2n})}\right\} \tag{4.16}$$

Set

$$\begin{aligned} S_I^\varepsilon(x, x'\,|\,\lambda) = \sum_{l=0}^{\infty}\lambda^l \int_{C_I}\cdots\int_{C_I} dy_1\cdots dy_l \\ \times\frac{\varepsilon e^{\varepsilon i(x-y_1)}}{2i(x-y_1)} h_I(y_1)\,L(y_1, y_2)\,h_I(y_2)\cdots \\ \times L(y_{l-1}, y_l)\,h_I(y_l)\,L(y_l, x'), \quad (\varepsilon=\pm) \end{aligned} \tag{4.17}$$

where the integration $\oint_{C_I} dy_\mu$ is along a simple closed C_I oriented clockwise, which encircle the points $a_1,\dots, a_{2m}$. In (4.17), x is supposed to be outside of C_I. We denote $\tilde{S}_I^\varepsilon(x, x'\,|\,\lambda)$ those obtained by letting x inside of C_I in (4.17). $S_I(x, x'\,|\,\lambda)$ is an entire function in both variables x, x'. $\tilde{S}_I^\varepsilon(x, x'\,|\,\lambda)$ is holomorphic except for a pole at $x=x'$. $S_I^\varepsilon(x, x'\,|\,\lambda)$ has branch points at $x=a_1,\dots, a_{2m}$. The singularity structure of $S_I(x, x'\,|\,\lambda)$ is a follows

$$S_I^\varepsilon(x, x'\,|\,\lambda) - \tilde{S}_I^\varepsilon(x, x'\,|\,\lambda) = \varepsilon\pi\lambda h_I(x)\,S_I(x, x'\,|\,\lambda) \tag{4.18}$$

We set

$$\begin{aligned} S_{\varepsilon,I}(x\,|\,\lambda) = \sum_{l=0}^{\infty}\lambda^l \int_{C_I}\cdots\int_{C_I} dy_1\cdots dy_l \\ \times L(x, y_1)\,h_I(y_1)\,L(y_1, y_2)\,h_I(y_2)\cdots \\ \times L(y_{l-1}, y_l)\,h_I(y_l)\,e^{\varepsilon i y_l}, \quad (\varepsilon=\pm) \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} S_{\varepsilon,I}^{\varepsilon'}(x\,|\,\lambda) = \sum_{l=0}^{\infty}\lambda^l \int_{C_I}\cdots\int_{C_I} dy_1\cdots dy_l \\ \times\frac{\varepsilon' e^{\varepsilon' i(x-y_1)}}{2i(x-y_1)} h_I(y_1)\,L(y_1, y_2)\,h_I(y_2)\cdots \\ \times L(y_{l-1}, y_l)\,h_I(y_l)\,e^{\varepsilon i y_l}, \quad (\varepsilon, \varepsilon'=\pm) \end{aligned} \tag{4.20}$$

In (4.20), x is supposed to be outside of C_I. We denote by $\tilde{S}^{\varepsilon'}_{\varepsilon,I}(x \mid \lambda)$ those obtained by letting x inside of C_I. The singularity structure of $S_{\varepsilon,I}(x, x' \mid \lambda)$ is as follows

$$S^{\varepsilon'}_{\varepsilon,I}(x \mid \lambda) - \tilde{S}^{\varepsilon'}_{\varepsilon,I}(x \mid \lambda) = \varepsilon' \pi \lambda h_I(x)\, S_{\varepsilon,I}(x \mid \lambda) \tag{4.21}$$

We define the matrices $Y_I(x)$, $\tilde{Y}_I(x)$ by

$$Y_I(x) = \begin{pmatrix} S_{+I}(x \mid \lambda) & S^{-}_{+I}(x \mid \lambda) \\ S_{-I}(x \mid \lambda) & S^{-}_{-I}(x \mid \lambda) \end{pmatrix}, \qquad \tilde{Y}_I(x) = \begin{pmatrix} S_{+I}(x \mid \lambda) & \tilde{S}^{-}_{+I}(x \mid \lambda) \\ S_{-I}(x \mid \lambda) & \tilde{S}^{-}_{-I}(x \mid \lambda) \end{pmatrix} \tag{4.22}$$

From the relation (4.19), we obtain the following monodromy properties

$$Y_I(x) = \tilde{Y}_I(x) \left\{ \frac{(x-a_1)(x-a_3)\cdots(x-a_{2m-1})}{(x-a_2)(x-a_4)\cdots(x-a_{2m})} \right\}^{\begin{pmatrix} 0 & \frac{i}{2}\lambda \\ 0 & 0 \end{pmatrix}} \tag{4.23}$$

The matrix $\tilde{Y}_I(x)$ is holomorphic and $\det \tilde{Y}_I(x) = 1$. It is known that $Y_I(x)$ satisfies the linear differential equation (4.49). See ref. 8. We define the matrices $Y_I^{(a,a')}(x)$ and $\tilde{Y}_I^{(a,a')}(x)$ by

$$Y_I^{(a,a')}(x) = (x-a)(x-a') \begin{pmatrix} S_I(x, a \mid \lambda) & S_I^{-}(x, a \mid \lambda) \\ S_I(x, a' \mid \lambda) & S_I^{-}(x, a' \mid \lambda) \end{pmatrix} \tag{4.24}$$

$$\tilde{Y}_I^{(a,a')}(x) = \begin{pmatrix} S_I(x, a \mid \lambda) & (x-a)(x-a')\, \tilde{S}_I^{-}(x, a \mid \lambda) \\ S_I(x, a' \mid \lambda) & (x-a)(x-a')\, \tilde{S}_I^{-}(x, a' \mid \lambda) \end{pmatrix} \tag{4.25}$$

From (4.17), we obtain the following formula

$$\begin{aligned} Y_I^{(a,a')}(x) &= \tilde{Y}_I^{(a,a')}(x) \{(x-a)(x-a')\}^{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}} \\ &\quad \times \left\{ \frac{(x-a_1)(x-a_3)\cdots(x-a_{2m-1})}{(x-a_2)(x-a_4)\cdots(x-a_{2m})} \right\}^{\begin{pmatrix} 0 & \frac{i}{2}\lambda \\ 0 & 0 \end{pmatrix}} \end{aligned} \tag{4.26}$$

The matrix $\tilde{Y}_I^{(a,a')}(x)$ is holomorphic and

$$\det \tilde{Y}_I^{(a,a')}(x) = \frac{a-a'}{2i} S_I(a, a' \mid \lambda) \tag{4.27}$$

Define the matrix $\tilde{Y}^{(a,a')}_{I\infty}(x)$ by

$$\tilde{Y}^{(a,a')}_{I\infty}(x)=\tilde{Y}^{(a,a')}_{I}(x)\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$$
$$=(x-a)(x-a')\begin{pmatrix}S_I^+(x,a\mid\lambda) & S_I^-(x,a\mid\lambda)\\ S_I^+(x,a'\mid\lambda) & S_I^-(x,a'\mid\lambda)\end{pmatrix} \qquad (4.28)$$

The matrix $Y^{(a,a')}_{I,\infty}(x)$ has the following local expansion at $x=\infty$

$$Y^{(a,a')}_{I,\infty}(x)=\left(S^{(a,a')}_{I,\infty}+O\left(\frac{1}{x}\right)\right)x\exp\left\{x\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\right\} \qquad (4.29)$$

where

$$S^{(a,a')}_{I,\infty}=\begin{pmatrix}S_{-I}(a\mid\lambda) & S_{+I}(a\mid\lambda)\\ S_{-I}(a'\mid\lambda) & S_{+I}(a'\mid\lambda)\end{pmatrix}\begin{pmatrix}-\frac{i}{2} & 0\\ 0 & \frac{i}{2}\end{pmatrix} \qquad (4.30)$$

We set

$$Z^{(a,a')}_{I}(x)=S^{(a,a')^{-1}}_{I,\infty}Y^{(a,a')}_{I}(x),\ \tilde{Z}^{(a,a')}_{I}(x)=S^{(a,a')^{-1}}_{I,\infty}\tilde{Y}^{(a,a')}_{I}(x) \qquad (4.31)$$

$Z^{(a,a')}_{I}(x)$ is so normalized that the local expansion at $x=\infty$ takes the form

$$Z^{(a,a')}_{I,\infty}(x)=\left(1+O\left(\frac{1}{x}\right)\right)x\exp\left\{x\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\right\} \qquad (4.32)$$

Here we start to consider our problem for correlation functions. Let $0\leqslant x'_1\leqslant\cdots\leqslant x'_n<\infty$, $0\leqslant x''_1\leqslant\cdots\leqslant x''_n<\infty$. Let I_p the union of n intervals $I_p=[x_1,x_2]\cup\cdots\cup[x_{2n-1},x_{2n}]$, where $0\leqslant x_1\leqslant\cdots\leqslant x_{2n}<\infty$ is the re-ordering of $x'_1,\ldots,x'_1,\ldots,x''_n$. Set $I_n=[-x_{2n},-x_{2n-1}]\cup\cdots\cup[-x_2,-x_1]$. In the sequel, we consider the case $m=2n$, $a_1=-x_{2n},\ldots,a_{2n}=-x_1$, $a_{2n+1}=x_1,\ldots,a_{4n}=x_{2n}$. We set $I=I_p\cup I_n$.

Lemma 4.1. The resolvent kernel has the following symmetries

$$S^{\varepsilon}_{I_p\cup I_n}(x,-x'\mid\lambda)=S^{-\varepsilon}_{I_p\cup I_n}(-x,x'\mid\lambda)$$
$$\tilde{S}^{\varepsilon}_{I_p\cup I_n}(x,-x'\mid\lambda)=\tilde{S}^{-\varepsilon}_{I_p\cup I_n}(-x,x'\mid\lambda) \qquad (4.33)$$

$$S_{\varepsilon,I_p\cup I_n}(-x\mid\lambda)=S_{-\varepsilon,I_p\cup I_n}(x\mid\lambda) \qquad (4.34)$$

$$S^{\varepsilon'}_{\varepsilon,I_p\cup I_n}(-x'\mid\lambda)=S^{-\varepsilon'}_{-\varepsilon,I_p\cup I_n}(x,\lambda)$$
$$\tilde{S}^{\varepsilon'}_{\varepsilon,I_p\cup I_n}(-x\mid\lambda)=\tilde{S}^{-\varepsilon'}_{-\varepsilon,I_p\cup I_n}(x\mid\lambda) \qquad (4.35)$$

The following is the key lemma.

Lemma 4.2. The resolvent kernel has the following linear relation

$$R_{\varepsilon, I_p}(x, x' \mid \lambda) = S_{I_p \cup I_n}(x, x' \mid \lambda) + \varepsilon S_{I_p \cup I_n}(x, -x' \mid \lambda) \qquad (\varepsilon = \pm) \tag{4.36}$$

Proof. The following characteristic relation holds,

$$\begin{aligned} S_{I_p \cup I_n}(x, x' \mid \lambda) + \lambda \int_{I_p} \{ & S_{I_p \cup I_n}(x, y \mid \lambda)\, L(y, x') \\ & + S_{I_p \cup I_n}(x, -y \mid \lambda)\, L(-y, x') \}\, dy = L(x, x') \end{aligned} \tag{4.37}$$

From (4.37) and the relation $\varepsilon^2 = 1$, we derive the following characteristic relation,

$$\begin{aligned} & S_{I_p \cup I_n(x, x' \mid \lambda) + \varepsilon S}{}^{I_p \cup I_n}(x, -x' \mid \lambda) \\ & \quad + \lambda \int_{I_p} (S_{I_p \cup I_n}(x, y \mid \lambda) + \varepsilon S_{I_p \cup I_n}(x, -y \mid \lambda))(L(y, x') + \varepsilon L(y, -x'))\, dy \\ & = L(x, x') + \varepsilon L(x, -x') \end{aligned} \tag{4.38}$$

This means the Eq. (4.36). ∎

Let us derive a formula for $d \log \det(1 - \lambda \hat{K}_{\varepsilon, I_p})$.

Proposition 4.3. We set $\omega_{\varepsilon, I_p}(\lambda) = d \log \det(1 - \lambda \hat{K}_{\varepsilon, I_p})$. Then we have

$$\begin{aligned} \omega_{\varepsilon, I_p}(\lambda) &= \operatorname{trace}\left(\sum_{j=1}^{2n} \tilde{Y}_{I_p \cup I_n}(x_j)^{-1} \frac{\partial}{\partial x} \tilde{Y}_{I_p \cup I_n}(x) \bigg|_{x = x_j} \begin{pmatrix} 0 & \lambda_j \\ 0 & 0 \end{pmatrix} dx_j \right) \\ & \quad - \varepsilon \frac{1}{2} \operatorname{trace}\left(\sum_{j=1}^{2n} \tilde{Y}_{I_p \cup I_n}(x_j)^{-1}\, \tilde{Y}_{I_p \cup I_n}(-x_j) \begin{pmatrix} 0 & \lambda_j \\ 0 & 0 \end{pmatrix} \frac{dx_j}{x_j} \right) \end{aligned} \tag{4.39}$$

$$\begin{aligned} &= \operatorname{trace}\left(\sum_{\delta = \pm}\ \sum_{1 \leqslant j < k \leqslant 2n} \delta \lambda_j \lambda_k A(x_j)\, A(\delta x_k)\, d \log(x_j - \delta x_k) \right) \\ & \quad - \varepsilon \operatorname{trace}\left(\sum_{j=1}^{2n} \lambda_j A(x_j) \left\{ A_\infty\, dx_j + \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{dx_j}{x_j} \right.\right. \\ & \quad \left.\left. - \lambda_j \frac{1}{2} A(-x_j) \frac{dx_j}{x_j} \right\} \right) \end{aligned} \tag{4.40}$$

Here the matrix $A(x_j)$ is defined by

$$A(x_j) = \tilde{Y}_{I_p \cup I_n}(x_j) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{Y}_{I_p \cup I_n}(x_j)^{-1}, \qquad \lambda_j = (-1)^{j+1} \frac{i\lambda}{2} \tag{4.41}$$

Proof. It is easy to see that

$$\frac{\partial}{\partial x_j} \log \det(1 - \lambda \hat{K}_{\varepsilon, I_p}) = (-1)^{j+1} \lambda R_{\varepsilon, I_p}(x_j, x_j \mid \lambda) \tag{4.42}$$

$$= (-1)^{j+1} \lambda \{ S_{I_p \cup I_n}(x_j, x_j) + \varepsilon S_{I_p \cup I_n}(x_j, -x_j) \} \tag{4.43}$$

From the definition, we can derive the following formula

$$\det \begin{pmatrix} S_{+I}(x) & S_{+I})x'- \\ S_{-I}(x) & S_{-I}(x') \end{pmatrix} = 2i(x - x')\, S_I(x, x') \tag{4.44}$$

Using this formula, we obtain

$$S_I(x, x) = \frac{i}{2} \det \begin{pmatrix} S_{+I}(x) & \dfrac{\partial}{\partial x} S_{+I}(x) \\ S_{-I}(x) & \dfrac{\partial}{\partial x} S_{-I}(x) \end{pmatrix} \tag{4.45}$$

$$= \frac{i}{2} \operatorname{trace} \left(\tilde{Y}_I(x)^{-1} \frac{\partial}{\partial x} \tilde{Y}_I(x) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \tag{4.46}$$

$$S_I(x, -x) = \frac{1}{4ix} \det \begin{pmatrix} S_{-I}(x) & S_{+I}(+x) \\ S_{-I}(x) & S_{-I}(-x) \end{pmatrix} \tag{4.47}$$

$$= \frac{1}{4ix} \operatorname{trace} \left(\tilde{Y}_I(x)^{-1}\, \tilde{Y}_I(-x) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \tag{4.48}$$

Hence we have the first line (4.39). Substituting (4.22) into the differential equation,

$$dY_I(x)\, Y_I(x)^{-1} = \sum_{j=1}^{2m} \lambda_j A(a_j)\, d \log(x - a_j) + A_\infty\, dx \tag{4.49}$$

which was derived in ref. 8, and comparing the coefficients of dx at $x=x_j$, we obtain the following

$$\frac{\partial}{\partial x}\tilde{Y}_{I_p\cup I_n}(x)\Big|_{x=x_j}\tilde{Y}_{I_p\cup I_n}(x_j)^{-1}$$
$$=A_\infty+\sum_{\varepsilon=\pm}\sum_{\substack{k=1\\k\neq j}}^{2n}\frac{\varepsilon(-1)^{j+1}}{x_j-\varepsilon x_k}A(\varepsilon x_k)+\frac{(-1)^j}{2x_j}A(-x_j)$$
$$-\tilde{Y}_{I_p\cup I_n}(x_j)\left\{\sum_{\varepsilon=\pm}\sum_{\substack{k=1\\k\neq j}}^{2n}\frac{\varepsilon\lambda_k}{x_j-\varepsilon x_k}L_k-\frac{\lambda_j}{2x_j}L_j\right\}\tilde{Y}_{I_p\cup I_n}(x_j) \tag{4.50}$$

where

$$A_\infty=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}$$

Substituting this relation into the first line (4.39), we obtain the second line (4.40). ∎

Remark. It is known that the matrices $A(x_j)$ are solutions of the generalized fifth Painlevé equations introduced in ref. 8.

Let us derive a formula for

$$d\log\det\left(1-\lambda\hat{K}_{\varepsilon,I_p}\,\Big|\,\begin{matrix}y\\y'\end{matrix}\right)$$

Let $-\infty<y,\ y'<+\infty$, $(y\neq y')$. In the sequal, we distinguish the following four cases.

1. $y,\ y'\neq x_1,\ldots,x_{2n}$.
2. $y'\neq x_1,\ldots,x_{2n},\ y=x_j$ for some j.
3. $y\neq x_1\ldots,x_{2n},\ y'=x_{j'}$ for some j'.
4. $y=x_j,\ y'=x_{j'}$ for some distinct j,j'.

Set

1. $J(y,y')=\{0,1,\ldots,2n+1\}$.
2. $J(y,y')=\{0,1,\ldots,2n+1\}\backslash\{j\}$.
3. $J(y,y')=\{0,1,\ldots,2n+1\}\backslash\{j'\}$.
4. $J(y,y')=\{0,1,\ldots,2n+1\}\backslash\{j,j'\}$.

Here we set $x_0=y$, $x_{2n+1}=y'$. Set $K(y,y')=J(y,y')\backslash\{0,2n+1\}$. Set the notations as follows

$$M_j=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad (j=\pm1,\dots,\pm 2n),\qquad M_0=M_{2n+1}=\begin{pmatrix}1&0\\0&0\end{pmatrix} \tag{4.51}$$

$$\lambda^{(y,y')}=(-1)^{j+1}\lambda\frac{i}{2}(y-x_j)(y'-x_j)$$

$$\lambda_{-j}^{(y,y')}=(-1)^{j}\lambda\frac{i}{2}(y+x_j)(y'+x_j)\qquad (j=1,\dots,2n), \tag{4.52}$$

$$\lambda_0^{(y,y')}=\lambda_{2n+1}^{(y,y')}=1$$

Let us state the Proposition.

Proposition 4.4. We set

$$\omega_{\varepsilon,I_p}^{(y,y')}(\lambda)=d\log\det\left(1-\lambda\hat{K}_{\varepsilon,I_p}\,\middle|\,\begin{matrix}y\\y'\end{matrix}\right)$$

We denote by d the exterior differentiation with respect to x_j $(j\in J(y,y'))$. Then we have

$$\omega_{\varepsilon,I_p}^{(y,y')}(\lambda)=\sum_{\delta=\pm}\delta\left(1+\varepsilon\frac{y-\delta y'}{y+\delta y'}\Delta^{(y,\delta y')}\right)^{-1}(\Omega_1^{(y,\delta y')}-\varepsilon\Omega_2^{(y,-\delta y')}-dy-dy') \tag{4.53}$$

Here we set

$$\Omega_1^{(y,y')}=\operatorname{trace}\left(\sum_{j\in J(y,y')}\lambda_j^{(y,y')}\tilde{Z}_{I_p\cup I_n}^{(y,y')}(x_j)^{-1}\frac{\partial}{\partial x}\tilde{Z}_{I_p\cup I_n}^{(y,y')}(x)\bigg|_{x=x_j}M_j\,dx_j\right) \tag{4.54}$$

$$\begin{aligned}&=\operatorname{trace}\Bigg(\sum_{\substack{i,j\in J(y,y')\\ i<j}}B_i^{(y,y')}B_j^{(y,y')}\,d\log(x_i-x_j)\\&\quad+\sum_{i\in J(y,y')}\sum_{j\in K(y,y')}B_i^{(y,y')}B_{-j}^{(y,y')}\frac{1}{x_i+x_j}dx_i+\sum_{i\in J(y,y')}B_i^{(y,y')}A_\infty\,dx_i\Bigg)\end{aligned} \tag{4.55}$$

$$\Omega_2^{(y,y')}=\operatorname{trace}\left(\sum_{j\in K(y,y')}\frac{i}{2}\lambda_j^{(y,y')}\tilde{Z}_{I_p\cup I_n}^{(y,y')}(x_j)^{-1}\tilde{Z}_{I_p\cup I_n}^{(y,y')}(-x_j)\,M_j\frac{dx_j}{x_j}\right) \tag{4.56}$$

$$\begin{aligned}&=-\Delta^{(y,-y)'}\operatorname{trace}\Bigg\{\sum_{j\in K(y,y')}(S_{I_p\cup I_n,\infty}^{(-y,y')}B_j^{(-y,y')}S_{I_p\cup I_n,\infty}^{(-y,y')^{-1}}\\&\quad+S_{I_p\cup I_n,\infty}^{(y,-y')}B_j^{(y,-y')}S_{I_p\cup I_n}^{(y,-y')^{-1}})\,M_0\frac{dx_j}{x_j}\Bigg\}\end{aligned} \tag{4.57}$$

where

$$\Delta^{(y,y')} = \det(S^{(y,-y')^{-1}}_{I_p \cup I_n, \infty} S^{(y,y')}_{I_p \cup I_n, \infty}) \tag{4.58}$$

$$B_j^{(y,y')} = \lambda_j^{(y,y')} \tilde{Z}_{I_p \cup I_n}(x_j)\, M_j \tilde{Z}_{I_p \cup I_n}(x_j)^{-1}, \quad (j = \pm 1, \dots \pm 2n, 0, 2n+1) \tag{4.59}$$

Proof. It is easy to see that

$$\frac{\partial}{\partial x_j} \log\det\left(1 - \lambda \hat{K}_{\varepsilon, I_p} \,\middle|\, \begin{matrix} y \\ y' \end{matrix}\right) = (-1)^{j+1} \lambda \frac{R_{\varepsilon, I_p}\left(\begin{matrix} y & x_j \\ y' & x \end{matrix} \,\middle|\, \lambda\right)}{R_{\varepsilon, I_p}(y, y' \mid \lambda)} (j \neq 0, 2n+1) \tag{4.60}$$

Then Lemma 4.2 and the following imply the jth part of (4.54)

$$S_{I_p \cup I_n}\left(\begin{matrix} y & x_j \\ y' & x_k \end{matrix} \,\middle|\, \lambda\right) = \frac{(y - x_j)(y' - x_k)}{(y - y')(x_j - x_k)} S_{I_p \cup I_n}\left(\begin{matrix} y & y' \\ x_j & x_k \end{matrix} \,\middle|\, \lambda\right) \tag{4.61}$$

For $j = 0,\ 2n+1$, the following imply the jth part of (4.56)

$$\frac{\partial}{\partial y} \log\det\left(1 - \lambda \hat{K}_{\varepsilon, I_p} \,\middle|\, \begin{matrix} y \\ y' \end{matrix}\right) = \frac{\partial}{\partial y} \log R_{\varepsilon, I_p}(y, y' \mid \lambda) \tag{4.62}$$

The second lines (4.55), (4.58) follows from the first ones by the same argument as in Theorem 4.3. ▮

Remarks. It is known that the matrices $B_j^{(y,y')}$ and $S^{(-y,y')}_{I_p \cup I_n, \infty} B_j^{(-y,y')} S^{(-y,y')^{-1}}_{I_p \cup I_n, \infty}$ are solutions of the generalized fifth Painlevé equations in ref. 8. For special cases $y = x_i$, $y' = x_j$, we have the following formula $\Delta^{(y,y')}$;

$$\frac{x_i - x_j}{x_i + x_j} \Delta^{(x_i, x_j)} = \frac{\operatorname{trace}(A(x_i)\, A(-x_j))}{\operatorname{trace}\left(A(x_i)\, A(x_j) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)} \tag{4.63}$$

Finally we give a proof of Theorem 2.3.

Proof of Theorem 2.3. Use the following formula

$$R_{\varepsilon, I_p}\left(\begin{matrix} y_1 \cdots y_k \\ y'_1 \cdots y'_k \end{matrix}\right) = \frac{\det\left(1 - \lambda \hat{K}_{\varepsilon, I_p} \,\middle|\, \begin{matrix} y_1 \cdots y_k \\ y'_1 \cdots y'_k \end{matrix}\right)}{-\lambda)^k \det(1 - \lambda \hat{K}_{\varepsilon, I_p})} \tag{4.64}$$

and apply Proposition 4.3 and Proposition 4.4. ▮

For $n=1$ and $0=x'<x$ case, because $R_{[0,x]}(0, x \mid \lambda)=2S_{[-x,x]}(0, x \mid \lambda)$, the differential equation becomes simpler form

$$\frac{d}{dx}\log \rho_1(0\ |x|+)$$
$$=\frac{\partial}{\partial y}\log S_{[-x,x]}(0, y \mid \lambda)\bigg|_{y=x}$$
$$=\operatorname{trace}\left(\left\{\left(B_0(0, -x, x)+\frac{1}{2}B_1(0, -x, x)\right)\frac{1}{x}+A_\infty\right\}B_0(0, -x, x)\right)-1 \tag{4.65}$$

Here

$$A_\infty=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

The 2×2 matrixes $B_j=B_j(a_0, a_1, a_2)$, $(j=0, 1, 2)$ depend on three parameters a_0, a_1, a_2 and satisfy the following differential systems that have the singularities at $y=a_0, a_1, a_2, \infty$. We denote by d the exterior differentiation with respect to y, a_0, a_1, a_2

$$dZ^{(a_0,a_2)}_{[a_1,a_2]}(y)=(B_0 d\log(y-a_0)+B_1 d\log(y-a_1)$$
$$+B_2 d\log(y-a_2)+A_\infty\, dy)\, Z^{(a_0,a_2)}_{[a_1,a_2]}(y) \tag{4.66}$$

where the 2×2 matrices $Z^{(b_1,b_2)}_{[b_3,b_4]}(y)$ are defined in (4.31). The integrability condition

$$d(dZ^{(a_0,a_2)}_{[a_0,a_2]}(y)Z^{(a_0,a_2)}_{[a_0,a_2]}(y)^{-1})$$
$$=dZ^{(a_1,a_2)}_{[a_0,a_2]}(y)Z^{(a_1,a_2)}_{[a_1,a_2]}(y)^{-1}\wedge Z^{(a_1,a_2)}_{[a_0,a_2]}(y)Z^{(a_1,a_2)}_{[a_0,a_2]}(y)^{-1} \tag{4.67}$$

gives rise to the following closed differential equation

$$dB_i=-\sum_{\substack{j=0\\ j\neq i}}^{2}[B_i, B_j]\, d\log(a_i-a_j)-[B_i, A_\infty]\, da_i, \qquad (i=0, 1, 2) \tag{4.68}$$

The eigenvalues of B_0, B_2 is $(0, 1)$. The eigenvalues of B_1 is $(0, 0)$. The diagonal of $B_0+B_1+B_2$ is $(1, 1)$. From the above matrix properties, we reduce (4.68) to the Hamiltonian Eqs. (2.30), (2.31), and (2.32) which was introduced in ref. 8. And we have the Eq. (2.29).

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